Signal Processing 93 (2013) 23–34

Contents lists available at SciVerse ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Low-complexity optimal discrete-rate spectrum balancing in digital subscriber lines

Martin Wolkerstorfer [a,*], Joakim Jaldén [b], Tomas Nordström [a,c]

[a] FTW Telecommunications Research Center Vienna, Donau-City-Straße 1, A-1220 Vienna, Austria
[b] ACCESS Linnaeus Center, Signal Processing Lab, KTH Royal Institute of Technology, Oskulds väg 10, SE 100 44 Stockholm, Sweden
[c] Centre for Research on Embedded Systems (CERES), Halmstad University, Box 823, SE-30118 Halmstad, Sweden

ARTICLE INFO

Article history:
Received 5 October 2011
Received in revised form
1 June 2012
Accepted 8 June 2012
Available online 18 June 2012

Keywords:
Power control
DSL
Interference channel
Combinatorial search

ABSTRACT

Discrete-rate spectrum balancing in interference-limited multi-user and multi-carrier digital subscriber lines (DSL) is a large-scale, non-convex and combinatorial problem. Previously proposed algorithms for its (dual) optimal solution are only applicable for networks with few users, while the suboptimality of less complex bit-loading algorithms has not been adequately studied so far. We deploy constrained optimization techniques as well as problem-specific branch-and-bound and search-space reduction methods, which for the first time give a low-complexity guarantee of optimality in certain multi-user DSL networks of practical size. Simulation results precisely quantify the suboptimality of multi-user bit-loading schemes in a thousand ADSL2 scenarios under measured channel data.

1. Introduction

We study the dynamic spectrum management (DSM) problem of *optimal* power allocation in interference-limited multi-user and multi-carrier digital subscriber line (DSL) systems with a finite set of transmission rates per subcarrier. Recent scalable approaches for optimal DSM are based on a Lagrange relaxation (LR) of the coupling sum-rate and sum-power constraints, resulting in independently solvable per-subcarrier problems. We propose novel branch-and-bound (BnB) and search-space reduction techniques which reduce the complexity of optimally solving these combinatorial per-subcarrier power control problems. Our optimization objective subsumes weighted sum-rate maximization and sum-power minimization as special cases. We emphasize that the studied combinatorial power control problem and our discrete search techniques are also of interest in other application domains where interference plays a role, such as in multi-user wireless networks [1–3].

A dual optimal multi-user DSM algorithm with discrete power levels was first proposed in [4], where an exhaustive search was applied to solve the per-subcarrier problems. In [5] we proposed a robust modification where the exhaustive search was made more efficient by avoiding the evaluation of infeasible rate combinations. A BnB algorithm for optimal per-subcarrier solutions was proposed in [6], although the BnB search had an exponential complexity in terms of time and memory, and asymmetric DSL (ADSL) downstream scenarios were only simulated for up to eight users. Optimal *continuous* power allocation has been studied for example in [7–10], where in [7] dual optimal results were presented for up to eight users in very high speed DSL (VDSL) scenarios. A traditional approach for discrete-rate DSM is discrete bit-loading (DBL) [11]. Various optimal DBL algorithms have been proposed in the single-user case, see [12,13] and references therein. However, only a few heuristics have been described in the multi-user case [14–16]. To the best of our knowledge, no study has up to now investigated the precise suboptimality of such multi-user bit-loading schemes in a larger set of scenarios.

* Corresponding author. Tel.: +43 1 5052830 27; fax: +43 1 505283099.
E-mail addresses: wolkerstorfer@ftw.at (M. Wolkerstorfer), jalden@kth.se (J. Jaldén), nordstrom@ftw.at, Tomas.Nordstrom@hh.se (T. Nordström).

0165-1684/$ - see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.sigpro.2012.06.007

Our main contribution is a low-complexity *optimal* discrete-rate allocation method for the per-subcarrier power control problems, which can be integrated in existing LR based multi-carrier DSM schemes. It consists of problem-specific implementations of two mechanisms which are generic parts of a modern nonlinear discrete problem solver [17,18]: a BnB framework and a variable-range reduction technique. After introducing the system and optimization model in Section 2 we propose in Section 3 two BnB schemes which show favorable computational and memory complexity to that previously proposed in [6]. The key feature of our BnB schemes is that maximum bit-loading information is passed between neighboring nodes in the BnB search-tree in order to reduce the number of infeasible rate evaluations. In Section 4 we suggest a low-complexity *optimal* search-space reduction (SSR) scheme based on a partly interference-free convex relaxation of the per-subcarrier problems. In the context of the related integer programming literature [17,19–21] our SSR scheme can be interpreted as a low-complexity, relaxation specific, nonlinear, objective-based [19] variable range reduction technique. Our SSR scheme is seen to be most effective in scenarios with low levels of crosstalk, e.g., low-bandwidth DSL systems or DSM problems with low target sum-rates. Simulation results are presented in Section 5, where we motivate our problem-specific approach by a comparison to a general-purpose solver for mixed-integer non-convex problems [18], investigate the effectiveness of the proposed techniques individually, and demonstrate the effect of target sum-rates on the solution complexity in a 16 user ADSL2 scenario. Furthermore, we analyze the sum-rate performance of a classical greedy multi-user DBL algorithm [14] for which we provide for the first time precise suboptimality figures in a thousand ADSL2 networks using measured channel data from [22]. Conclusions of this work are drawn in Section 6.

2. Problem formulation

2.1. System model

We consider an interference-limited multi-carrier DSL system with U lines and C subcarriers, and denote the sets of indices for users and subcarriers by $\mathcal{U}=\{1,\dots,U\}$ and $\mathcal{C}=\{1,\dots,C\}$, respectively. We denote by p_u^c the power spectral density (PSD), by N_u^c the received noise spectral density, and by H_{uu}^c and H_{ui}^c the squared magnitudes of the channel coefficient of user u and from user i to user u, respectively, on subcarrier c. We use the gap-approximation in [11] and write the achievable rate per symbol for user $u\in\mathcal{U}$ on subcarrier $c\in\mathcal{C}$ as

$$r_u^c(\mathbf{p}^c)=\log_2\left(1+\frac{H_{uu}^c p_u^c}{\Gamma(\sum_{i\in\mathcal{U}\setminus u}H_{ui}^c p_i^c+N_u^c)}\right), \tag{1}$$

where $\mathbf{p}^c=[p_1^c,\dots,p_U^c]^T$ and where Γ is the SNR-gap. We will compactly write all users' rates as $\mathbf{r}^c(\mathbf{p}^c)=[r_1^c(\mathbf{p}^c),\dots,r_U^c(\mathbf{p}^c)]^T$ and use $\mathbf{p}^c(\mathbf{r}^c)$ to denote the unique [23] power allocation resulting in the rate vector $\mathbf{r}^c$. This power $\mathbf{p}^c(\mathbf{r}^c)$ can be computed as the solution of a matrix equation [4] where the matrix as well as the right-hand-side vector depend on $\mathbf{r}^c$. Considering a regulatory power spectral mask $\hat{p}_u^c$ and a bit-cap $\hat{\theta}$ we can express the feasible set of PSD's on subcarrier c which yield discrete rates as

$$\mathcal{Q}^c=\{\mathbf{p}^c\,|\,r_u^c(\mathbf{p}^c)\in\mathcal{B},\ p_u^c\in[0,\hat{p}_u^c],\ u\in\mathcal{U}\}, \tag{2}$$

where $\mathcal{B}=\{0,\theta,\dots,\hat{\theta}\}$ is the set of possible bit-allocations per subcarrier and user. Correspondingly we define the set of *all* users' possible rates per subcarrier as $\mathcal{L}=\times_{u\in\mathcal{U}}\mathcal{B}$, the set of *feasible* rates by $\mathcal{Q}_{\mathbf{r}}^c=\{\mathbf{r}^c\in\mathcal{L}\,|\,\mathbf{p}^c(\mathbf{r}^c)\in\mathcal{Q}^c\}$, and the set of *infeasible* rates by $\overline{\mathcal{Q}}_{\mathbf{r}}^c=\{\mathbf{r}^c\in\mathcal{L}\,|\,\mathbf{r}^c\notin\mathcal{Q}_{\mathbf{r}}^c\}$. Note that this work can be readily extended to more general (finite) sets $\mathcal{B}$.

2.2. Primal optimization problem

We focus on a generic U user DSM problem with target sum-rates $\mathbf{R}\in\mathcal{R}_+^U$ and maximum sum-power $\mathbf{P}\in\mathcal{R}_+^U$ given by [4,6,24]

$$P^*_{(\mathbf{R},\mathbf{P})}=\underset{\mathbf{p}^c\in\mathcal{Q}^c,c\in\mathcal{C}}{\text{minimize}}\quad \sum_{c\in\mathcal{C}}f(\mathbf{p}^c,\hat{\mathbf{w}},\check{\mathbf{w}}) \tag{3a}$$

$$\text{subject to}\quad \sum_{c\in\mathcal{C}}\mathbf{r}^c(\mathbf{p}^c)\succcurlyeq\mathbf{R}, \tag{3b}$$

$$\sum_{c\in\mathcal{C}}\mathbf{p}^c\preceq\mathbf{P}, \tag{3c}$$

where the objective is a weighted-sum defined by

$$f^c(\mathbf{p}^c,\hat{\mathbf{w}},\check{\mathbf{w}})=\hat{\mathbf{w}}^T\mathbf{p}^c-\check{\mathbf{w}}^T\mathbf{r}^c(\mathbf{p}^c),\quad c\in\mathcal{C}, \tag{4}$$

where $\hat{\mathbf{w}},\check{\mathbf{w}}\in\mathcal{R}_+^U$ are constant weights. Special cases of this objective are the users' sum-rate or sum-power.

2.3. Dual optimization problem

The classical approach for dealing with the constraints in (3b) and (3c) for systems with a large number of subcarriers C is Lagrange relaxation [25] of these constraints, resulting in the dual problem to (3) defined as (cf. [4])

$$D^*_{(\mathbf{R},\mathbf{P})}=\underset{\boldsymbol{\lambda}\succcurlyeq\mathbf{0},\boldsymbol{\nu}\succcurlyeq\mathbf{0}}{\text{maximize}}\ q^{\text{tot}}(\boldsymbol{\lambda},\boldsymbol{\nu}), \tag{5}$$

where $\boldsymbol{\lambda},\boldsymbol{\nu}\in\mathcal{R}^U$ are the Lagrange multipliers associated with constraints (3b) and (3c), respectively, and where the dual function is defined as [25]

$$q^{\text{tot}}(\boldsymbol{\lambda},\boldsymbol{\nu})=\sum_{c\in\mathcal{C}}q^c(\boldsymbol{\lambda},\boldsymbol{\nu})+\boldsymbol{\lambda}^T\mathbf{R}-\boldsymbol{\nu}^T\mathbf{P}, \tag{6}$$

$$q^c(\boldsymbol{\lambda},\boldsymbol{\nu})=\min_{\mathbf{p}^c\in\mathcal{Q}^c}\{f^c(\mathbf{p}^c,\hat{\mathbf{w}}+\boldsymbol{\nu},\check{\mathbf{w}}+\boldsymbol{\lambda})\},\quad\forall c\in\mathcal{C}. \tag{7}$$

Our contribution is the proposal of optimal methods for solving the decomposable dual subproblems in (7) and the analysis of the suboptimality of greedy multi-user bit-loading [14] using the proposed optimal methods. The optimal dual objective $D^*_{(\mathbf{R},\mathbf{P})}$ lower-bounds the optimal primal objective $P^*_{(\mathbf{R},\mathbf{P})}$ in (3a) as given by the weak duality inequality [25] $D^*_{(\mathbf{R},\mathbf{P})}\le P^*_{(\mathbf{R},\mathbf{P})}$. This relation will be used in the analysis of the suboptimality of two greedy multi-user bit-loading algorithms in Section 5.5, namely the multi-user scheme in [14] and single-user bit-loading considering the worst-case

Algorithm 1. DFB tree search.

$[\mathbf{r}^*, f^*, f^{lb}] = \text{DFBSearch}(\mathbf{r}^0, \boldsymbol{\lambda}, \boldsymbol{\nu}, \Phi)$
. . . .
Initialization— bEnd = false, $\mathbf{r}=\mathbf{0}$, $u=1$, bRet=false, $\hat{\mathbf{r}}=\mathbf{1}\hat{\theta}$,
$\mathbf{p}^{\min}=\mathbf{0}$, $lb=-\infty$, $\hat{\mathbf{r}}^{\text{tmp}}=\hat{\mathbf{r}}$, $\mathbf{B}^{\text{tmp}}=\mathbf{B}=[\mathbf{0},\hat{\mathbf{r}}]$, $f^{lb}=\infty$,
if $\mathbf{0}\preceq\mathbf{p}(\mathbf{r}^0)\preceq\hat{p}$ and $f(\mathbf{p}(\mathbf{r}^0),\hat{\mathbf{w}}+\boldsymbol{\nu},\check{\mathbf{w}}+\boldsymbol{\lambda})<0$ **then**
  Set incumbent $\mathbf{r}^*=\mathbf{r}^0$, $f^*=f(\mathbf{p}(\mathbf{r}^0),\hat{\mathbf{w}}+\boldsymbol{\nu},\check{\mathbf{w}}+\boldsymbol{\lambda})$
else Set incumbent $\mathbf{r}^*=\mathbf{0}$, $f^*=0$
$[\mathbf{B}_{u+1:U,:},\mathbf{p}^{\min},lb^{SSR}]=SSR(\boldsymbol{\lambda},\boldsymbol{\nu},\mathbf{r},\hat{\mathbf{r}}^{\text{tmp}},0,f^*)$
while $\neg$bEnd
. . . .
  Start search at $\mathbf{r}=\mathbf{B}_{:,1}$ —***Update Allocation***
  if bRet **then** Set $r_u=r_u+\theta$ and bRet=false
    if $r_u>\min\{\hat{r}_u^{\text{tmp}},\mathbf{B}_{u,2}^{\text{tmp}}\}$ **then** bRet=true **else**
      if $\mathbf{0}\preceq\mathbf{p}(\mathbf{r})\preceq\hat{p}$ **then** Set $\mathbf{p}^{\min}=\mathbf{p}(\mathbf{r})$
        if $f(\mathbf{p}(\mathbf{r}),\hat{\mathbf{w}}+\boldsymbol{\nu},\check{\mathbf{w}}+\boldsymbol{\lambda})<f^*$ **then**
          $\mathbf{r}^*=\mathbf{r}$, $f^*=f(\mathbf{p}(\mathbf{r}^*),\hat{\mathbf{w}}+\boldsymbol{\nu},\check{\mathbf{w}}+\boldsymbol{\lambda})$
      else bRet=true, $\hat{r}_u^{\text{tmp}}=r_u-\theta$
. . . .
   if $\neg$ bRet **then** —***Bounding***
    $[\mathbf{B}_{u+1:U,:}^{\text{tmp}},\mathbf{p}^{\min},lb^{SSR}]=SSR(\boldsymbol{\lambda},\boldsymbol{\nu},\mathbf{r},\hat{\mathbf{r}}^{\text{tmp}},u,f^*)$
    if $u<U$ **then**

$$r_i^{\max}=\begin{cases} r_i, & 1\le i\le u,\\ \min\{\hat{r}_i^{\text{tmp}},\mathbf{B}_{i,2}^{\text{tmp}}\}, & u+1\le i\le U\end{cases}$$

     $lb=l^{\mathbf{r}^{(u)}}(\mathbf{p}^{\min},\mathbf{r}^{\max})$, cf. (10), $lb=\max\{lb,lb^{SSR}\}$
. . . .
     if $lb\ge\min\{f^*,\Phi\}$ **then** —***Branching***
      Set bRet=true, $f^{lb}=\min\{f^{lb},lb\}$
     else $u=u+1$
   else Set $r_u=\mathbf{B}_{u,1}$, $u=u-1$
    if $u<U$ **then** $\hat{r}_{u+1}=\mathbf{B}_{u+1,2}$
    if $u=0$ **then** bEnd=true
if $f^*\le\Phi$ **then** $f^{lb}=f^*$

Algorithm 2. Search-space reduction scheme.

$[\mathbf{B}_{u+1:U,:}^{\text{tmp}},\mathbf{p}^{\min},\check{L}]=SSR(\boldsymbol{\lambda},\boldsymbol{\nu},\mathbf{r}^{\text{init}},\hat{\mathbf{r}}^{\text{tmp}},u,f^*)$
Initialize $\mathbf{p}=\mathbf{p}(\mathbf{r}^{\text{init}})$, $\mathbf{p}^{\min}=\mathbf{p}$, $\overline{\mathbf{p}}=\mathbf{p}$, $\overline{\mathbf{r}}=\mathbf{r}^{\text{init}}$
. . . .
Find the discrete optimum without crosstalk :
Compute $\tilde{\mathbf{N}}\in\mathcal{R}^U$, $\tilde{N}_i=N_i+\sum_{1\le j\le u}H_{ij}p_j,\forall i\in\mathcal{U}$
Repeat $\forall i\in\mathcal{U}\backslash\{1,\ldots,u\}$ Equations (B.2)–(B.4)
$\check{L}=(\hat{\mathbf{w}}+\boldsymbol{\nu})^T\overline{\mathbf{p}}-(\check{\mathbf{w}}+\boldsymbol{\lambda})^T\overline{\mathbf{r}}$, Set $\mathbf{r}=\overline{\mathbf{r}}$
while $(\mathbf{p}(\mathbf{r})\notin\mathcal{Q})$ **do** $r_i=\max\{0,r_i-\theta\},u<i\le U$
$L^{\text{target}}=\min\{f^*,f(\mathbf{p}(\mathbf{r}),\hat{\mathbf{w}}+\boldsymbol{\nu},\check{\mathbf{w}}+\boldsymbol{\lambda})\}$
. . . .
Find the restricted search-region around $\overline{\mathbf{r}}$:
 for $i=u+1,\ldots,U$ **do**
  Set $\mathbf{r}=\overline{\mathbf{r}}$, **Increase** r_i in steps of θ **until** $r_i>\hat{r}_i^{\text{tmp}}$
  or $(\hat{\mathbf{w}}+\boldsymbol{\nu})^T\mathbf{p}-(\check{\mathbf{w}}+\boldsymbol{\lambda})^T\mathbf{r}>L^{\text{target}}$, with
  $p_i=p_i(\mathbf{r}^{\text{init}}),i\in[1,u],p_i=p_i^{\text{tmp}}(r_i)$ as in (B.3), $i\in\mathcal{U}\backslash[1,u]$,
  $\mathbf{B}_{i,2}^{\text{tmp}}=r_i-\theta$, $\mathbf{r}=\overline{\mathbf{r}}$
  Decrease r_i in steps of θ **until** $r_i<0$ **or**
  $(\hat{\mathbf{w}}+\boldsymbol{\nu})^T\mathbf{p}-(\check{\mathbf{w}}+\boldsymbol{\lambda})^T\mathbf{r}>L^{\text{target}}$, with
  $p_i=p_i(\mathbf{r}^{\text{init}}),i\in[1,u],p_i=p_i^{\text{tmp}}(r_i)$ as in (B.3), $i\in\mathcal{U}\backslash[1,u]$,
  $\mathbf{B}_{i,1}^{\text{tmp}}=r_i+\theta$, $p_i^{\min}=p_i^{\text{tmp}}(r_i+\theta)$

crosstalk based on the spectral mask constraints. Complementing our simulation study on the suboptimality of greedy bit-loading in the multi-user case, the following theorem establishes primal optimality of greedy bit-loading in the special case of a single user.

Theorem 1. *For* $U=1$ *greedy bit-loading is optimal and it holds that* $P^*_{(\mathbf{R},\mathbf{P})}=D^*_{(\mathbf{R},\mathbf{P})}$, $\forall\mathbf{R}\in\{\tilde{\mathbf{R}}\in\mathcal{R}^U\,|\,\tilde{R}_u=k\cdot\theta,k\in\mathcal{Z}_+,\forall u\in\mathcal{U}\}$.

See Appendix A for a proof. Theorem 1 extends a previous result on the optimality of single-user bit-loading [12] to a more general objective.

We will now study the problem in (7) in the general multi-user case and drop the subcarrier index c throughout the rest of the paper for ease of notation. Also, to facilitate an intuitive understanding of the algorithms proposed in the following we rephrase the problem in (7) in terms of rates as

$$\underset{\mathbf{r}\in\mathcal{Q}_{\mathbf{r}}}{\text{minimize}}\, f(\mathbf{p}(\mathbf{r}),\hat{\mathbf{w}}+\boldsymbol{\nu},\check{\mathbf{w}}+\boldsymbol{\lambda}). \tag{8}$$

The complete proposed method for solving the problem in (8), which will be described in the Sections 3 and 4, can be found in detail in Algorithms 1 and 2. It consists of two interrelated parts, where the first one is a technique to reduce the search space $\mathcal{Q}_{\mathbf{r}}$ by reasoning based on a fast solvable convex problem relaxation. The second part is a branch-and-bound search scheme which serves to efficiently explore the reduced search space and to return a provably optimal solution to the problem in (8). We emphasize that our contribution does not depend on the specific algorithm which generates the Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ and is therefore also applicable to previous work on DSM algorithms. However, for our simulations in Section 5 we use the finitely converging spectrum balancing framework described in [24]. It is based on a linear problem (LP) which is iteratively updated using the per-subcarrier solutions of the problems in (8). The dual solution of this LP comprises the Lagrange multipliers which are used to define the subproblems in (8).

3. Branch-and-bound (BnB) for discrete bit and power allocation

Solving the per-subcarrier problem in (8) can be regarded as a sequence of U consecutive decisions, each assigning a certain number of bits to one of the U users. This can be illustrated in form of a decision tree, where decisions are made starting with that of user 1 (cf. Fig. 1). A node at tree-depth u corresponds to a vector of bit-allocations $\mathbf{r}^{(u)}\in\mathcal{R}^u$ of *"already loaded users"* 1 up to u. The feasible set of rates $\mathcal{Q}_{\mathbf{r}}$ corresponds to a subset of all leaf-nodes $\mathcal{L}$ at the bottom of the search-tree. Branch-and-bound (BnB) [26] is a systematic and exact search method for finding the leaf-node $\mathbf{r}^{(U)}\in\mathcal{Q}_{\mathbf{r}}\subseteq\mathcal{L}$ with minimum objective value $f(\mathbf{p}(\mathbf{r}^{(U)}),\hat{\mathbf{w}}+\boldsymbol{\nu},\check{\mathbf{w}}+\boldsymbol{\lambda})$, i.e., for solving the subproblem in (8). The algorithm starts at the root node of the tree (cf. Fig. 1) and at each node makes a branching decision, i.e., decides which node to explore next at the next-lower tree-level. The second key component of the method beside the branching rule is the computation of lower-bounds on the objective values of any leaf-node belonging to a subtree rooted at node $\mathbf{r}^{(u)}$, cf. Section 3.1. However, the *exact* computation of the tightest possible

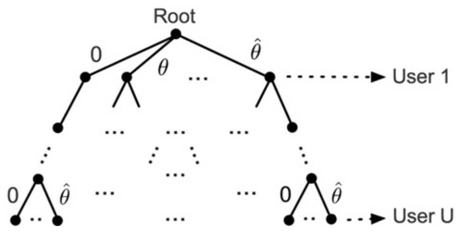

Fig. 1. Illustration of the search-tree associated with a per-subcarrier problem in (7).

lower bound would necessitate the solution of the problem (cf. (1), (2), (4), and (8))

$$\underset{\substack{r_i \in \{0,\theta,\ldots,\hat{\theta}\}, u+1 \le i \le U, \\ p_i \in [0,\hat{p}_i], \forall i \in \mathcal{U}}}{\text{minimize}} \sum_{\{i \in \mathcal{U} | i \le u\}} ((\hat{w}_i+\nu_i)p_i - (\check{w}_i+\lambda_i) r_i^{(u)}) + \sum_{\{i \in \mathcal{U} | i \ge (u+1)\}} ((\hat{w}_i+\nu_i)p_i - (\check{w}_i+\lambda_i) r_i) \tag{9a}$$

$$\text{subject to} \quad r_i^{(u)} \le r_i(\mathbf{p}), \quad 1 \le i \le u, \tag{9b}$$

$$r_i \le r_i(\mathbf{p}), \quad u+1 \le i \le U, \tag{9c}$$

which differs from the original problem in (8) only in the fact that the rates of users $i, i \le u$, are fixed at $\mathbf{r}^{(u)}$. Consequently, solving (8) with a BnB algorithm using the tightest lower bound would be as costly as solving (8) by a brute-force enumeration of $\mathcal{Q}_\mathbf{r}$. However, the efficiency of the BnB algorithm comes from the use of efficiently computable lower-bounds to the optimal objective of the problem in (9), which can be used to infer the suboptimality of a subtree ("pruning" the subtree) based on the comparison of this lower-bound with the objective value of the best ("incumbent") solution $\mathbf{r}^{(U)} \in \mathcal{Q}_\mathbf{r}$ found so far. The algorithm only stops when it has either visited or pruned all leaves of the tree. Hence, there is a complexity tradeoff in BnB schemes between the computation of lower-bounds and the exploration of the tree.

3.1. *Computing lower-bounds in BnB schemes*

Before proposing explicit BnB methods we will describe the computation of bounds which are less complex than the solution of the problem in (9). We generically denote such a lower-bound for the subtree rooted at $\mathbf{r}^{(u)}$ as

$$l^{\mathbf{r}^{(u)}}(\mathbf{p}^{\min}, \mathbf{r}^{\max}) = (\hat{\mathbf{w}}+\boldsymbol{\nu})^T \mathbf{p}^{\min} - (\check{\mathbf{w}}+\boldsymbol{\lambda})^T \mathbf{r}^{\max}, \tag{10}$$

where $\mathbf{p}^{\min}$ and $\mathbf{r}^{\max}$ are chosen appropriately, cf. [6] for a specific BnB scheme dependent choice. A selection which gives a valid lower-bound in (10) to the exact lower bound in (9) is $p_i^{\min} = p_i(\mathbf{r}^{(u)})$, $r_i^{\max} = r_i^{(u)}$, $1 \le i \le u$, and $p_i^{\min} = 0$, $r_i^{\max} = \hat{\theta}$, $(u+1) \le i \le U$, where $p_i(\mathbf{r}^{(u)}) = p_i(\mathbf{r})$, $i \in \mathcal{U}$, as defined in (1) with $\mathbf{r} \in \mathcal{R}^U$ being formed by extending $\mathbf{r}^{(u)}$ with $(U-u)$ zeros. This selection results in a valid lower-bound as (a) the rates of already loaded users are fixed while those of the remaining users are constrained by $\hat{\theta}$; (b) the power needed to support the rates $\mathbf{r}^{(u)}$ in (9b) monotonously increases with the crosstalk noise and is therefore the lowest when the power of the remaining users is the lowest possible (i.e., 0); and (c) the objective in (9a) is monotonously increasing in the powers and decreasing in the rates. In Section 3.2.1 and 4 we will see two less conservative ways of obtaining bounds by modifying $p_i^{\min}$ and $r_i^{\max}$ for users i, where $(u+1) \le i \le U$.

Another means of efficiently lower-bounding $f(\mathbf{p}(\mathbf{r}), \hat{\mathbf{w}}+\boldsymbol{\nu}, \check{\mathbf{w}}+\boldsymbol{\lambda})$ for all leaf nodes $\mathbf{r}$ in the subtree rooted at $\mathbf{r}^{(u)}$ also derives from the above key observation that the power needed to support the rate $r_i(\mathbf{p})$ in (9b) and (9c) without interference among users lower-bounds the corresponding values with interference. A relaxation of the problem in (9) in this respect can be formulated as

$$\underset{\substack{r_i \in \{0,\theta,\ldots,\hat{\theta}\}, \\ p_i \in [0,\hat{p}_i], u+1 \le i \le U}}{\text{minimize}} \sum_{\{i \in \mathcal{U} | i \le u\}} ((\hat{w}_i+\nu_i)p_i(\mathbf{r}^{(u)}) - (\check{w}_i+\lambda_i) r_i^{(u)}) + \sum_{\{i \in \mathcal{U} | i \ge (u+1)\}} ((\hat{w}_i+\nu_i)p_i - (\check{w}_i+\lambda_i) r_i) \tag{11a}$$

$$\text{subject to} \quad r_i \le \log_2\left(1+\frac{H_{ii}p_i}{\Gamma \tilde{N}_i}\right), \quad u+1 \le i \le U, \tag{11b}$$

which differs from the exact lower-bounding problem in (9) in that the powers p_i for users i, $i \le u$, are fixed at the above mentioned lowest values $p_i(\mathbf{r}^{(u)})$ and the constraints in (9b) are neglected. Furthermore, the total received noise in the rate functions $r_i(\mathbf{p})$ in (9c) is lower-bounded by considering only the interference from already loaded users, which by the same argument as above leads to the lower-bound on the received noise given by $\tilde{N}_i = N_i + \sum_{1 \le j \le u} H_{ij} p_j(\mathbf{r}^{(u)})$, for the remaining users $i, (u+1) \le i \le U$. Solving the problem in (11) forms the basis of our search-space reduction algorithm in Section 4 and its solution with linear complexity in U is detailed in Appendix B.

Yet another relaxation of (9) in the form of a linear program (LP) can be obtained in that we first apply a continuous relaxation of the discrete rate variables, and relax (9c) as in (11b). This relaxation would truly capture the interference induced to the already loaded users through the constraints in (9b). However, we find that the improvements of the pruning process in our BnB schemes are outweighed by its complexity and we will therefore not considered it further.

3.2. *Depth-first BnB (DFB) and best-first BnB (BFB) search*

The symmetric BnB scheme in [6], which we refer to as the regular splitting based BnB (RSB), branches by expanding all subtrees in parallel. This leads to exponential worst-case memory requirements, making RSB inapplicable for a larger number of users. The second disadvantage, as the simulations in Section 5.2 will reveal, is that testing infeasible allocations $\mathbf{r} \in \overline{\mathcal{Q}}_\mathbf{r}$ significantly contributes to the search complexity of this method. The BnB schemes presented in the following remedy exactly these two disadvantages by having linear worst-case memory requirements and introducing an explicit mechanism to exploit the impact of some users' bit allocation on other users.

3.2.1. Depth-first BnB

Depth-first BnB (DFB) explores the search-tree starting from the node $\mathbf{r}^{(U)} = \mathbf{0}$, corresponding to the bottom-left node in Fig. 1. Bits are iteratively increased, starting with user U corresponding to the bottom level of the search-tree. Algorithm 1 summarizes the DFB scheme in more detail, with specific aspects being explained in the remainder of this section. In lines 2–5 we initialize the needed data structures and set the incumbent solution based on the initial allocation $\mathbf{r}^0$ passed as input to the function. Lines 8–14 and 19–24 implement the searching strategy (i.e., the update of the currently investigated node in the search tree), with an update of the minimal power allocation $\mathbf{p}^{\min}$ in line 11 as used in (10), and a conditional update of the incumbent objective f^* in line 13. In line 19 the subtree is pruned based on the comparison of the current lower bound lb to the current upper bounds. In lines 6 and 15–18 we perform further tasks needed to improve the lower bound in (10), including an update of $\mathbf{r}^{\max}$, and a call of Algorithm 2 to be specified in Section 4.

Information regarding the maximum feasible[1] bit-loading of user u, in a subtree rooted at level $u-1$, can be used to produce an upper bound on user u's feasible bit-loadings in all neighboring subtrees rooted at level $u-1$. This is made possible by the intuitive observation that increasing the rate of any other user $i \in \mathcal{U}$, $i \neq u$, cannot increase the maximum feasible rate of user u as increasing the rate of another user can only increase the overall interference.

The basic idea is exemplified in Fig. 2, where once node n_a has been found infeasible it immediately follows that also node n_b and n_c must also be infeasible. Consequently, having visited node n_a in the DFB search over the tree illustrated in Fig. 2 and found this node infeasible, the upper bound on the maximum rate of user u can be set to $r_u^{\max} = \theta$ when searching the neighboring subtrees rooted at level $u-1$. In the general case, having found a node $\mathbf{r}^{(u)}$ infeasible, the DFB can set $r_u^{\max} = r_u^{(u)} - \theta$ in the search of the remaining neighboring subtrees at level $u-1$, i.e., in any remaining subtree rooted at $\mathbf{r}^{(u-1)}$ where $r_i^{(u-1)} = r_i^{(u)}$ for $i = 1, \ldots, u-2$.

The value of the observation described above is that in order to test the general feasibility of a node $\mathbf{r}^{(u)}$ requires the computation of the unique power allocation $\mathbf{p}(\mathbf{r}^{(u)})$ as the solution of a set of linear equations (cf. Section 2.1). By using the upper bound on the rate of user u we can avoid testing certainly infeasible nodes, thus reducing the number of computations required in the BnB implementation. The procedure can be applied at any level of the tree, although the upper bound $r_u^{\max}$ must be re-set to $\hat{\theta}$ when the search back-tracks to level $u-2$.

[1] An arbitrary node $\mathbf{r}^{(u)}$ in the BnB search-tree is termed feasible if it can be extended to a feasible leaf-node $\mathbf{r} \in \mathcal{Q}_\mathbf{r}$ or, equivalently, a feasible bit allocation for all users. Note that to determine if a node $\mathbf{r}^{(u)}$ is feasible, it is sufficient to extend $\mathbf{r}^{(u)}$ to a leaf-node $\mathbf{r}^{(U)}$ along the all-zeros-path, such that $\mathbf{r}_i^{(U)} = \mathbf{r}_i^{(u)}$ for $i = 1, \ldots, u$ and $\mathbf{r}_i^{(U)} = 0$ for $i = u+1, \ldots, U$, and verify the feasibility of $\mathbf{r}^{(U)}$. If $\mathbf{r}^{(U)}$ is infeasible (i.e., $\mathbf{r}^{(U)} \notin \mathcal{Q}_\mathbf{r}$), no feasible extensions exist.

Besides lowering the number of feasibility tests performed (cf. the use of variable $\hat{r}_u^{\text{tmp}}$ in Algorithm 1), the reduction of $r_u^{\max}$ also strengthens the lower bound in (10), cf. lines 14 and 18 in Algorithm 1. Furthermore, the initialization of the incumbent in lines 3–5 can be further improved, e.g., by using heuristics [27] or the solutions of other DSM algorithms, cf. Section 5. Note that DFB as well as the scheme presented in the following section can make use of an available objective bound Φ to improve the pruning process, cf. line 19 in Algorithm 1 and [24] where the specific relaxation gives such bounds as a byproduct. The variable f^{lb} in Algorithm 1 serves as a certificate in case no feasible solution with objective lower than Φ is found, i.e., as a lower-bound for the objective of any (e.g., pruned) feasible allocation. In the following we analyze the complexity of DFB.

Corollary 1 (*Wolkerstorfer et al. [24, Thm. 2]*). *The complexity of the search scheme DFB in Algorithm 1 for solving the subproblem in (8) is polynomial in the number of users U given $H_{ui}/H_{uu} \geq \alpha > 0$, $\forall u \in \mathcal{U}$.*

Proof. While by [24, Thm. 2] we have that $|\mathcal{Q}_\mathbf{r}|$ grows polynomially in U, we also see that the same holds for the complexity per feasible allocation in the BnB Algorithm 1 (i.e., including the solution of a linear system for evaluating $\mathbf{p}(\mathbf{r})$ and the calculation of lower-bounds through (10) or the solution of the relaxation in (11) as detailed in Appendix B). Furthermore, the number of tested *infeasible* allocations $\mathbf{r} \in \overline{\mathcal{Q}}_\mathbf{r}$ can be bounded as follows: Assuming a feasible bit allocation (e.g., $\mathbf{r} = \mathbf{0}$), the algorithm proceeds by increasing the number of bit-steps for user U by one. If this is feasible we obtain another feasible bit allocation, while if it is not we return to the previous user $U-1$ and increase its bit allocation by one, setting that of user U to 0. Following this procedure we find that the maximum number of failed trials per feasible bit allocation is bounded by the depth of the search-tree U, concluding our argument. □

3.2.2. Best-first BnB

Best-first BnB (BFB) differs from DFB in that branching is not performed systematically starting from lower bit-allocations and proceeding to higher ones. More precisely, the algorithm branches the node which has the lowest lower-bound in (10). Whether this is a good decision will clearly depend on the quality of the lower-bound. Similarly as in DFB we can make use of the maximum bit-loading information $r_u^{\max}$. However, considering only subtrees rooted at level $u-1$ with equal bit-allocation $r_i^{(u-1)}$ for all previous levels i, $i \leq u-2$ (cf. Fig. 2), in BFB we might visit a subtree with higher rate $r_{u-1}^{(u-1)}$ before visiting a subtree with lower rate $r_{u-1}^{(u-1)}$ at level $u-1$. Therefore we might not have access to the closest estimate $r_u^{\max}$ we would have obtained if we had visited the neighboring subtree with lower rate $r_{u-1}^{(u-1)}$ first. This deteriorates the effectiveness of the bounds in (10) and the search as a whole. For brevity we omit a detailed algorithm description of BFB as we found BFB inferior compared to DFB for a larger number of users, cf. our results in Section 5.

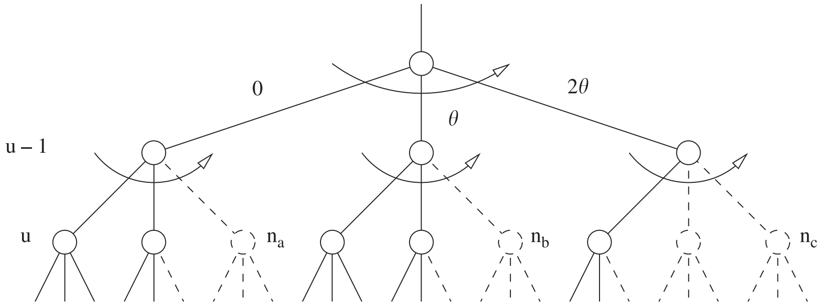

Fig. 2. Part of BnB search tree for the case of $\hat{\theta}=2\theta$ containing a set of three neighboring subtrees rooted at level $u-1$. Dashed lines and circles are used to illustrate infeasible nodes.

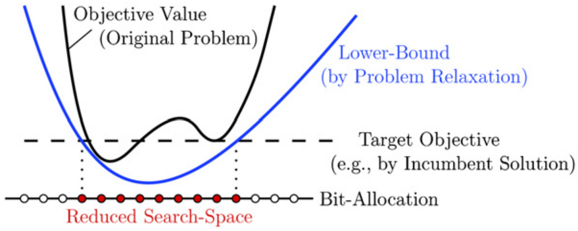

Fig. 3. Illustration of the idea behind objective-based search-space reduction (SSR).

4. A search-space reduction (SSR) scheme

In [24] an upper-bound on the size of the feasible search space $|\mathcal{Q}_{\mathbf{r}}|$ for the per-subcarrier problem in (8) was found which grows with decreasing crosstalk strength. The bound has therefore its highest value if we assume no crosstalk between users. This is however counter-intuitive in face of the fact that a greedy bit-loading algorithm would then solve (8) optimally, cf. the proof of Theorem 1 and the fact that the per-subcarrier problem decouples into single-user problems in the absence of interference as seen in Section 3.1. In the following we suggest a method which can take advantage of low-interference cases in order to reduce the size of the search-space for the exact problem in (9) corresponding to a subtree rooted at $\mathbf{r}^{(u)}$.

The idea behind our SSR scheme is to first solve the lower-bounding problem in (11) which partly neglects inter-user interference and therefore is a relaxation of the exact problem in (9). Then, starting from the optimum of the lower-bounding problem, we search the bit-allocations with a lower-bound (i.e., objective value in the lower-bounding problem) below the real objective (i.e., with fully considered inter-user interference) of a known feasible solution for the exact problem in (9). This idea is illustrated in Fig. 3 where it is shown how a lower bound on the objective provides a tightened superset of all bit-allocations that can improve upon the target objective. Correspondingly, the SSR scheme in Algorithm 2 contains two parts, where in the first one in lines 4–8 we solve the lower-bounding problem in (11) as described in Appendix B and search a feasible target objective value L^{target} for the exact problem in (9). Note that also the objective value of the current *global* incumbent solution can be used as the target objective value. While the incumbent objective might be lower than any feasible objective value in the subtree considered in the exact problem, the SSR scheme will not exclude any solution in the subtree from the search-space that has a better objective than the current incumbent and therefore supports the pruning of sub-optimal solutions. The second part of the SSR scheme is detailed in lines 10–14 of Algorithm 2 and concerns the search-space reduction. As seen in Appendix B the lower-bounding problem can be solved by separate problems for users i, $u+1\le i\le U$. Hence, also the minimum and maximum rate in the subtree rooted at $\mathbf{r}^{(u)}$ of user i can be found by searching, starting from the optimum of the lower-bounding problem, the minimum and maximum rates with lower-bound below the target objective value.

This SSR method can be employed at any level of the search-tree of BFB and DFB, cf. line 16 in Algorithm 1, but only on the root node in RSB. Note also that minimum power/maximum bit-loading information is obtained by SSR which is used for computing lower-bounds in line 18 of Algorithm 1.

5. Simulation results

5.1. Performance comparison to a general-purpose solver

In order to motivate our problem-specific BnB and variable-range reduction mechanisms DFB and SSR we compare them to the open-source general-purpose solver for non-convex mixed-integer problems "Couenne" [18] in terms of CPU time for various sum-rate maximization problems.[2] In order to have problem instances that are easier to solve by general purpose software we base the simulations on five user upstream transmission ADSL2 scenarios,[3] with topologies generated by uniformly

[2] Both algorithms solve, for each scenario, the per-subcarrier problem in (8) for all subcarriers $c\in\mathcal{C}$ only once, that is, for the set of weights and Lagrange multipliers that are output in the first iteration by the sum-rate maximizing master LP [24] after being initialized by the solution of the bit-loading scheme in [14]. For Couenne we apply default parameters, except for the branching priority of continuous variables which was set lower than for integer variables. The platform is Windows 7 on an Intel quad-core system running at 2.4 GHz with 4 GB of RAM.

[3] The parameters for ADSL2 and VDSL follow the corresponding standards in [28] (Annex B.1.3) and in [29] (band plan 997-M 1 x-M, a flat spectral mask constraint at −60 dBm/Hz, alien crosstalk according

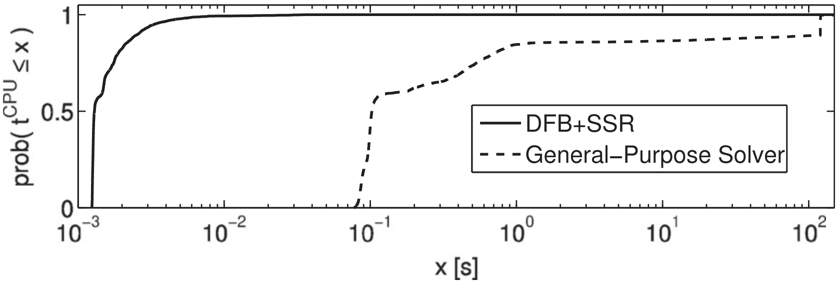

Fig. 4. Cumulative distribution function for the CPU time t^{CPU} of the proposed combined search mechanism (DBF and SSR) and an applicable general-purpose solver under a time-out of 2 min in randomly sampled five user ADSL2 scenarios.

sampling the users' line-lengths between 500 m and 2000 m. The used crosstalk channel data is based on measurements in [22] and a random cable selection as the number of measured lines exceeds U. Fig. 4 depicts the cumulative distributions of CPU time for our combined scheme (DFB and SSR) and for Couenne. The main observation from these results is that there is a subset of the per-subcarrier problems which could not be solved by Couenne in the set time-limit of 120 s. Hence, the development of problem-specific techniques for the problem at hand is qualitatively justified.

5.2. Comparison of BnB methods

In the following we investigate the average complexity of solving the subproblems in (8) under the three BnB schemes outlined in Section 3. In order to study a more challenging DSM case we use numerous VDSL upstream scenarios as described below and a 99% worst-case crosstalk model [11]. For tractability we restrict ourselves to the simulation of every 50th subcarrier out of the more than 1600 subcarriers under fixed values of weights $\check{w}_u = 0$, $w_u = 1/U$, and identical Lagrange multipliers $\lambda_u = \lambda$, $\nu_u = 0$, $\forall u \in \mathcal{U}$. To capture the performance of these methods exclusively we do not make use of incumbent initializations. Also, we avoid machine dependency of our performance evaluation to a large extent by focussing on two reproducible complexity merits: the number of visited feasible leaf nodes (rate allocations) $\mathbf{r} \in \mathcal{Q}_\mathbf{r}$ and the total number of visited leaf nodes $\mathbf{r} \in \mathcal{L}$ (corresponding to the number of solved matrix equations [4] in the power evaluation $\mathbf{p}(\mathbf{r})$ for bit allocations $\mathbf{r}$), where the latter also includes the tested allocations $\mathbf{r} \in \overline{\mathcal{Q}_\mathbf{r}}$ which turn out infeasible. Furthermore, we neglect the complexity of other logical operations needed to perform the BnB searches. As explained in Section 3.2 the RSB scheme has more severe memory requirements compared to our two BnB search proposals, and hence our comparisons favor this previously proposed scheme. The network scenarios are based on a set of specified line lengths $\{200, 400, 600, 800\}$ m, and forming all U-combinations with repetitions[4] to allocate users to these lengths, cf. [27] for details.

Fig. 5(a) and (b) show the total number of visited leaf nodes and the number of visited feasible leaf nodes, respectively, over the subcarrier index under different search schemes for six users. The bend of the curves at index 500 is due to the use of two non-adjacent frequency sub-bands. In general we find that the complexity in solving the per-subcarrier problems (8) decreases in all schemes with the subcarrier frequency due to the increasing crosstalk coupling per unit-length and channel attenuation. In a "naive search" all allocations in $\times_{u \in \mathcal{U}} \{0, \theta, \ldots, \hat{\theta}_u\} \subseteq \mathcal{L}$ are evaluated, where we define $\hat{\theta}_u$, $\hat{\theta}_u \leq \hat{\theta}$, as the maximum number of bits user u can transmit without interference, cf. Fig. 5(b). However, we see that the number of feasible leaf nodes $|\mathcal{Q}_\mathbf{r}|$ in the search-tree, labelled "all feasible", is far below this number. This complexity reduction can already be achieved by a modified exhaustive search [5, Algorithm 2]. Furthermore, we observe a reduction in the number of visited leaf nodes by the BnB schemes BFB, DFB and RSB, with DFB performing best over the whole range of subcarrier problems, cf. Fig. 5(b). This can be further explained by comparing Fig. 5(a) and (b), where we see that a large part of the complexity of RSB lies in the evaluation of *infeasible* allocations, which is avoided by our proposed BnB mechanism in Section 3.2.1. Regarding the comparison in average sum-complexity over subcarriers in Fig. 5(c) we see that for less than four users RSB performs better than DFB. However, for a higher number of users we observe a growing gap in complexity between the two schemes with DFB performing better, both, in terms of computational and memory complexity.

5.3. The impact of the Lagrange multipliers on the complexity

We use the same simulation setup as in the previous section and demonstrate in Fig. 6(a) the dependency of the average (over users and network scenarios) search complexity and the average achieved rates on the Lagrange multipliers $\lambda_u = \lambda$, $u \in \mathcal{U}$, associated with the

(*footnote continued*)
to ETSI VDSL noise A), with $\Gamma = 12.8$ dB and $\theta = 1$. The background noise for ADSL2 is set to $N_u^c = -120$ dBm/Hz, while that for VDSL is chosen as -140 dBm/Hz. The maximum bit-allocation $\hat{\theta}$ for the single-carrier simulations in Sections 5.1– 5.3 was set to 16 as it was used in the BnB simulations in [6], while for multi-carrier simulations we use a standard compliant value of 15.

[4] For example, for $U = 6$ the number of thereby generated scenarios is 84.

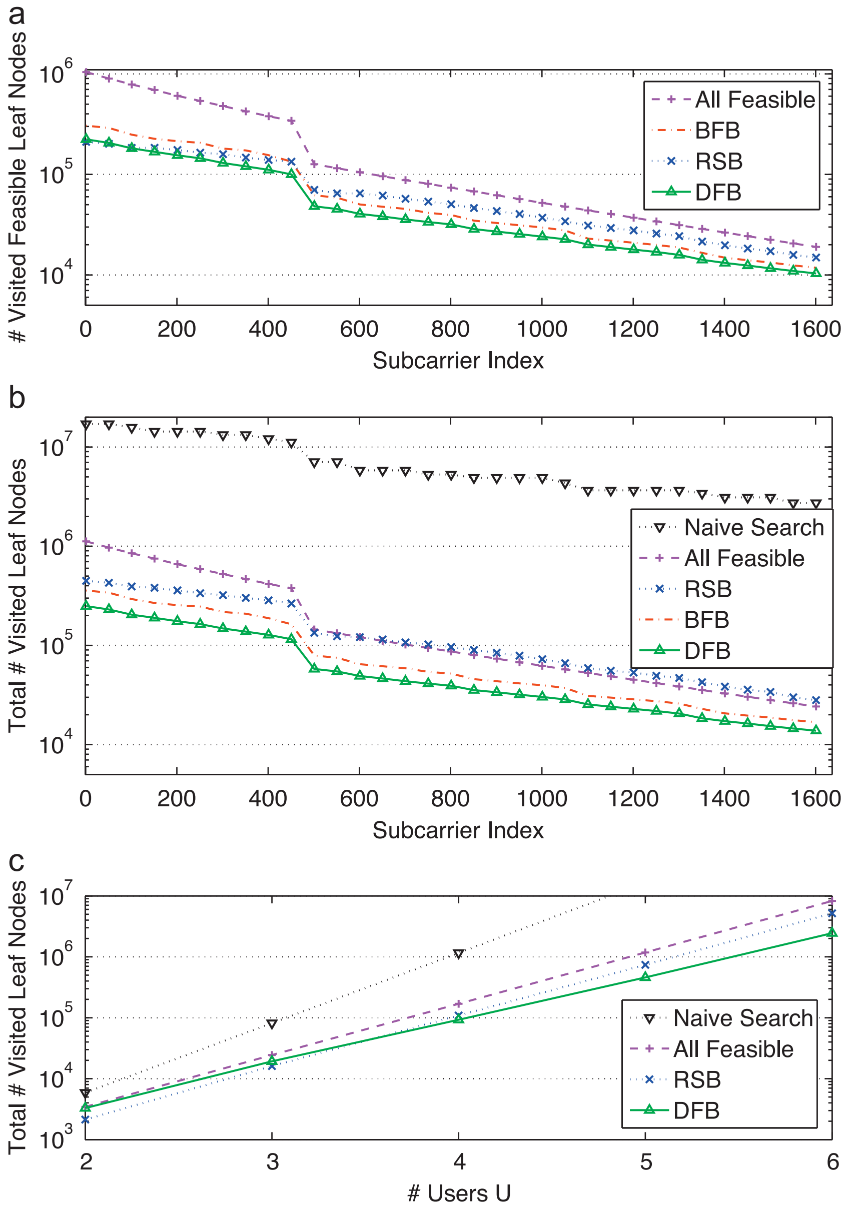

Fig. 5. Average complexity of branch-and-bound searches in terms of the number of total or feasible visited leaf nodes for solving the per-subcarrier problems in (8) with $\hat{w}_u = 1/U$, $\lambda_u = 10^{-3}$, $\forall u \in \mathcal{U}$; in (a) and (b) over the subcarrier index for $U=6$; in (c) the sum-complexity for all per-subcarrier problems over the number of users U.

rates. Our SSR scheme is shown to successfully reduce the search space for lower values of λ, where we note that the algorithm is only executed once at the root of the search-tree, cf. the curve labelled "DFB + SSR, No Bounding". Higher values of λ on the other hand lead to higher optimal per-subcarrier rates and therefore to stronger interference levels among users at the optimum. This on the other hand decreases the quality of the lower-bound computed from the interference-free problem in (11) and therefore the performance of the SSR scheme. However, DFB outperforms RSB for larger values of λ, making the joint application of SSR and DFB search outperform RSB over the whole range of λ, cf. the curve labelled "DFB+SSR, With Bounding". Furthermore, as explained in Section 3, RSB has a worst-case storage requirement that increases exponentially in the number of users, while that of DFB increases only linearly. Altogether we argue that DFB is the preferable BnB scheme, especially in scenarios with a larger number of users.

Another aspect in Fig. 6(a) is that the average search complexity of all shown BnB schemes only increases up to a certain value of λ, after which it decreases again. As the number of feasible allocations is independent of λ we attribute this behavior to the lower-bound $l^{\mathbf{r}^{(u)}}(\mathbf{p}^{\min},\mathbf{r}^{\max})$ in (10). We argue that during BnB search there is typically a good bound $\mathbf{r}^{\max}$, available, be it either based on the search-region definition as in RSB [6] or on message passing of maximum bit-loading information as in DFB

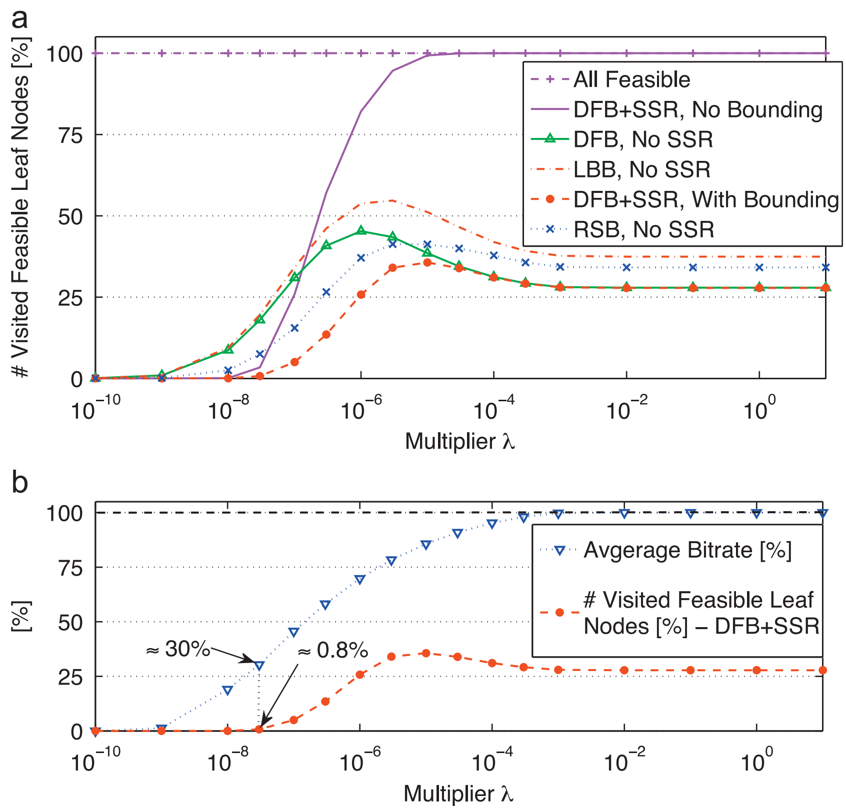

Fig. 6. Average branch-and-bound (BnB) performance in solving the problem in (8) over the Lagrange multiplier $\lambda_u = \lambda$, $\forall u \in \mathcal{U}$, in various six user VDSL scenarios. (a) Sum-complexity over all selected per-subcarrier problems. (b) Comparison between the complexity of our combined search method and the (optimal) achieved average per-user and per-subcarrier rates.

and BFB. However, lower-bounds $p_i^{\min}$ on the optimal power consumption of a user i, $u < i \le U$, for a subtree rooted at $\mathbf{r}^{(u)}$ are harder obtained, where notably our SSR scheme allows us to get lower-bounds other than $p_i^{\min} = 0$, cf. line 14 of Algorithm 2. Altogether we have stronger lower-bounds for either lower values of λ where the optimal power levels are small, or higher values of λ where the total weight $(\check{\mathbf{w}} + \boldsymbol{\lambda})$ on the rates is dominant.

The multiplier λ controls the rates' weight in the objective $f(\mathbf{p}, \hat{\mathbf{w}} + \boldsymbol{\nu}, \check{\mathbf{w}} + \boldsymbol{\lambda})$ defined in (4). Fig. 6(b) illustrates the average per-user rate over the multiplier λ in percent of the maximum achieved value (at $\lambda = 10$) and compares it to the complexity of our combined search method (DFB and SSR) taken from Fig. 6(a). We observe that a wide range of low average rates is achievable at fairly low complexity. For example, we find that for $\lambda = 3 \times 10^{-8}$ our combined search method (DFB and SSR) visits less than 0.8% of the total number of feasible leaf nodes in the search-tree, while the resulting average rate is more than 30% of the maximum one at $\lambda = 10$.

5.4. *The impact of target sum-rates on the complexity of DSM*

Differently to the previous section we will now directly analyze the impact of the target sum-rates on the complexity of a dual optimal DSM scheme as a whole in various sum-power minimization problems. The simulation is based on a single ADSL2 topology with 16 users transmitting in upstream direction and situated at $800, 850, \ldots, 1550$ m distance from the deployment point, and we use the average crosstalk per unit-length between all measured cables for the calculation of the crosstalk couplings. A maximum total number of evaluated feasible allocations of 10^9 was set to limit the simulation time. From the simulation results shown in Fig. 7 we find that the complexity of our dual-optimal DSM algorithm qualitatively increases with the target sum-rates. However, we also note that the sum-rate maximization (cf. the point at the bottom-right in Fig. 7) had again a complexity which was lower compared that of sum-power minimization at the highest shown target sum-rates. This is most likely due to the fact that sum-power constraints were not tight in this scenario and the multipliers $\boldsymbol{\nu}$ are consequently all zero. Therefore a good lower bound on the objective can be computed by knowing a good upper-bound on the number of bit-steps that can be loaded, as obtained in our DFB search, cf. Section 3.2.1. This observation is also in accordance with the interpretation of Fig. 6(a) in Section 5.3.

5.5. *Performance evaluation of greedy multi-user bit-loading*

We will next investigate the suboptimality in sum-rate maximization problems of two multi-user discrete bit-loading (DBL) approaches: the greedy multi-user DBL

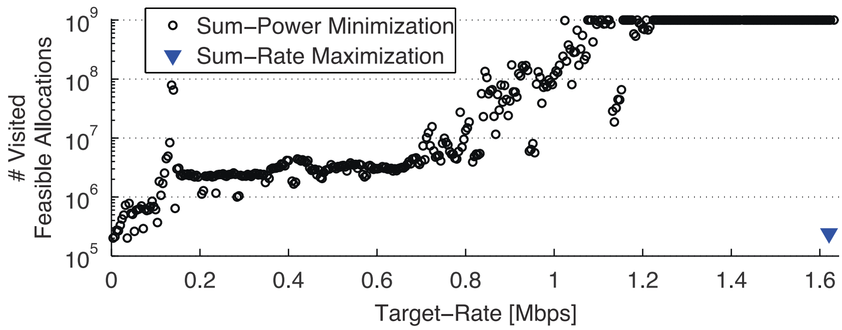

Fig. 7. Complexity of solving a dual sum-power minimization problem using the Lagrange multiplier update method in [24] and the optimal techniques DFB and SSR in a distributed 16 user ADSL2 scenario.

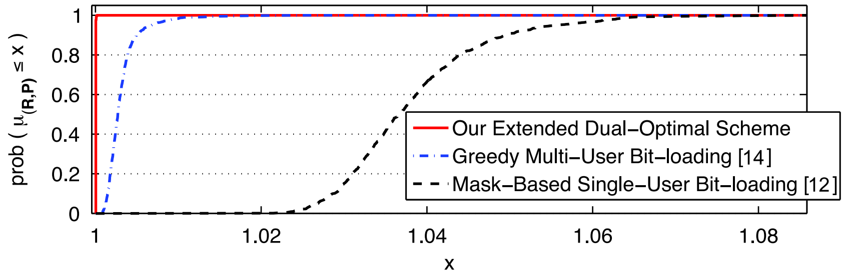

Fig. 8. Cumulative distribution function of the performance ratio $\mu_{(\mathbf{R},\mathbf{P})}$ in (12) for our extended dual-optimal DSM scheme and two multi-user bit-loading algorithms.

(MDBL) scheme in [14] and single-user bit-loading [12] under worst-case crosstalk noise computed using the users' PSD masks and the (measured) cross-channel data. The performance metric we consider for these two heuristics is their performance ratio defined as

$$\mu_{(\mathbf{R},\mathbf{P})} = \frac{D^*_{(\mathbf{R},\mathbf{P})}}{P^{\text{heur}}_{(\mathbf{R},\mathbf{P})}}, \tag{12}$$

where we denote the objective value of a heuristic bit-loading scheme achieved by a feasible solution of the primal problem in (3) as $P^{\text{heur}}_{(\mathbf{R},\mathbf{P})} \geq P^*_{(\mathbf{R},\mathbf{P})}$. For example, in the considered case of $\hat{\mathbf{w}} = \mathbf{0}$, meaning a pure sum-rate maximization problem, we have $D^*_{(\mathbf{R},\mathbf{P})} \leq P^{\text{heur}}_{(\mathbf{R},\mathbf{P})} \leq 0$ and it therefore holds that $\mu_{(\mathbf{R},\mathbf{P})} \geq 1$. The dual optimal objective $D^*_{(\mathbf{R},\mathbf{P})}$ in (5) is computed by our dual optimal DSM scheme, consisting of the optimal Lagrange multiplier update algorithm in [24], initialized at the solution under MDBL, and using our optimal techniques DFB and SSR for the subproblems in (8). In order to obtain a clearer picture of the duality gap $(P^*_{(\mathbf{R},\mathbf{P})} - D^*_{(\mathbf{R},\mathbf{P})})$ in realistic scenarios we also derive primal feasible solutions to the problem in (3) by applying the heuristic described in [24] on top of our dual optimal DSM scheme. We simulated 1000 ADSL2 scenarios with 10 users and other simulation parameters and a random topology and cable selection as described in Section 5.1.

Our results are shown in Fig. 8. Regarding the cumulative distribution[5] for our extended dual-optimal scheme we see that the performance ratio $\mu_{(\mathbf{R},\mathbf{P})}$ was in fact nearly 1 in all tested scenarios. More precisely, its suboptimality and therefore the duality gap is below 0.01% of $D^*_{(\mathbf{R},\mathbf{P})}$ in more than 99.6% of the scenarios, with a confidence of 99% according to a *t*-test. Similarly, the suboptimality of MDBL is guaranteed to lie below 1% in more than 97.7% of the scenarios, demonstrating the near-optimality of MDBL in realistic ADSL2 scenarios. Surprisingly even the conservative mask-based single-user DBL approach leads to a suboptimality of below 5% in more than 91.1% of the scenarios.

6. Conclusions

We propose an optimal low-complexity algorithm for the solution of a discrete-rate, multi-user, single-subcarrier power control problem. This algorithm is applicable in any Lagrange relaxation based dynamic spectrum management (DSM) scheme for interference-limited multi-carrier digital subscriber lines (DSL). Its components are a branch-and-bound (BnB) scheme with linear storage requirements and an efficient problem-specific search-space reduction (SSR) technique. Simulation results show a dependency of the per-subcarrier BnB search complexity with the rate-related Lagrange multipliers, confirming our intuition that the difficulty of solving energy-minimization problems increases with the target sum-rate by trend. The SSR scheme is based on a convex problem relaxation where interference is omitted based on the search-tree. Correspondingly, it is seen to be effective in the case of low target sum-rates, higher background noise, or low-bandwidth DSL systems

[5] In the computer science literature [30] this empirical distribution is referred to as "performance profile".

such as ADSL2. Under such a setting we demonstrate that DSM problems with twice as many users can be solved *optimally* compared to previously published numerical results. With our algorithm we were able to present optimal results in 1000 randomly generated 10-user ADSL2 scenarios, and to show that the average gap to the optimum of a heuristic multi-user bit-loading scheme is less than 1% in more than 97% of the scenarios.

Acknowledgements

This work was supported by Grant ICA08-0046 of the Swedish Foundation for Strategic Research (SSF). The Competence Center FTW Forschungszentrum Telekommunikation Wien GmbH is funded within the program COMET – Competence Centers for Excellent Technologies by BMVIT, BMWA, and the City of Vienna. The COMET program is managed by the FFG.

Appendix A. Proof of Theorem 1

Proof. We assume $U=1$ and hence all vectors in this appendix are in $\mathcal{R}_+$. First we prove the optimality of a greedy bit-loading (GBL) algorithm which, starting from $\mathbf{r}^c=0, \forall c \in \mathcal{C}$, sequentially loads θ bits where they incur the least extra cost $f^c_{\mathbf{r}^c+\theta}=w_u(\mathbf{p}^c(\mathbf{r}^c+\theta)-\mathbf{p}^c(\mathbf{r}^c))-\check{w}_u\theta$. We define the set of possible bit-steps $\mathcal{E}$, $|\mathcal{E}|=\sum_{c\in\mathcal{C}}|\mathcal{Q}^c_{\mathbf{r}}|$, a function which assigns each element of $\mathcal{E}$ the associated cost $f^c_{\mathbf{r}^c}$, $\mathbf{r}^c$ being the rate after the bit-step, and the matroid $(\mathcal{E},\mathcal{I})$, $\mathcal{I}\subseteq\mathcal{E}$. Optimality of GBL follows now from the optimality of greedily picking elements out of $\mathcal{E}$ [31, Sec. 7.5] and the monotonicity of $f^c_{\mathbf{r}^c}$ in $\mathbf{r}^c$. It follows that GBL demands the minimum sum-power for a given target sum-rate. With this in mind we denote the optimum of (3) for neglected sum-power constraints (3c) by $P^*(\mathbf{R})$. We denote the maximal achievable sum-rate by $\hat{\mathbf{R}}=\sum_{c\in\mathcal{C}}\max\{\mathbf{r}^c|\mathbf{r}^c\in\mathcal{Q}^c_{\mathbf{r}}\}$ and the minimum number of loaded bits after which all remaining possible bit-steps have a positive cost by $\check{\mathbf{R}}\in\mathcal{R}_+$, i.e., $\check{\mathbf{R}}=\min_{\mathbf{R}}\{\mathbf{R}|\exists\mathbf{r}^c\in\mathcal{Q}^c_{\mathbf{r}}, c\in\mathcal{C}, \sum_{c\in\mathcal{C}}\mathbf{r}^c\le\mathbf{R}, f^c_{\mathbf{r}^c+\theta}>0, \forall c\in\mathcal{C}\}$. By optimality we have that $P^*(\mathbf{R})$ is constant for $\mathbf{R}\le\min\{\check{\mathbf{R}},\hat{\mathbf{R}}\}$. We will therefore focus on the case $\check{\mathbf{R}}<\hat{\mathbf{R}}$, where $P^*(\mathbf{R})$ is strictly monotonously increasing in $\mathbf{R}$ for $\mathbf{R}>\check{\mathbf{R}}$ due to optimality of GBL. Convexity of $P^*(\mathbf{R})$ over $\mathbf{0}\le\mathbf{R}\le\hat{\mathbf{R}}$ follows from

$$\alpha P^*(\mathbf{R}-k_\alpha\theta)+\beta P^*(\mathbf{R}+k_\beta\theta) \tag{A.1a}$$

$$\ge \alpha(P^*(\mathbf{R})-k_\alpha\epsilon_\alpha)+\beta(P^*(\mathbf{R})+k_\beta\epsilon_\beta), \tag{A.1b}$$

$$= P^*(\mathbf{R})-\alpha k_\alpha\epsilon_\alpha+\beta k_\beta\epsilon_\beta \ge P^*(\mathbf{R}), \tag{A.1c}$$

where in (A.1a) we form a convex combination of any two target sum-rates in the given interval with lower and higher sum-rate than $\mathbf{R}$ (i.e., $k_\alpha,k_\beta\ge0$), respectively. In (A.1b) we use the optimality of GBL, defining $\epsilon_\alpha=P^*(\mathbf{R})-P^*(\mathbf{R}-\theta)$ and $\epsilon_\beta=P^*(\mathbf{R}+\theta)-P^*(\mathbf{R})$. In (A.1c) we use $\alpha+\beta=1$ and $\alpha k_\alpha=\beta k_\beta$ as we are interested in the convex combination at sum-rate $\mathbf{R}$, and again the optimality of GBL implying $\epsilon_\alpha\le\epsilon_\beta$. Next we regard any convex combination of feasible solutions to our primal problem in (3), which are represented by the three-dimensional set

$$\check{\mathcal{Q}}=\left\{\left[\sum_{c\in\mathcal{C}}f(\mathbf{p}^c,\hat{\mathbf{w}},\check{\mathbf{w}}),\sum_{c\in\mathcal{C}}\mathbf{r}^c(\mathbf{p}^c),\sum_{c\in\mathcal{C}}\mathbf{p}^c\right]^T\middle|\,\mathbf{p}^c\in\mathcal{Q}^c, c\in\mathcal{C}\right\}. \tag{A.2}$$

By Carathéodory's theorem [25, Prop. B.6], for $U=1$ any point in the convex hull of $\check{\mathcal{Q}}$ can be represented by the convex combination of at most four points in $\check{\mathcal{Q}}$. We define target sum-rates $\mathbf{R}\in\{\tilde{\mathbf{R}}|\tilde{\mathbf{R}}=k\theta, k\in\mathcal{Z}_+\}$ and pick any such combination with sum-power values $\mathbf{P}_\alpha,\mathbf{P}_\beta,\mathbf{P}_\gamma,\mathbf{P}_\delta$, with achieved sum-rates $\mathbf{R}_\alpha,\mathbf{R}_\beta$, $\mathbf{R}_\gamma$, $\mathbf{R}_\delta$, and with a set of weights $\alpha,\beta,\gamma,\delta\ge0$, such that $\alpha\mathbf{R}_\alpha+\beta\mathbf{R}_\beta+\gamma\mathbf{R}_\gamma+\delta\mathbf{R}_\delta=\mathbf{R}$ and $\alpha\mathbf{P}_\alpha+\beta\mathbf{P}_\beta+\gamma\mathbf{P}_\gamma+\delta\mathbf{P}_\delta\le\mathbf{P}$. The resulting objective value of the combination is given as, cf. (3a) and (4),

$$\hat{\mathbf{w}}^T(\alpha\mathbf{P}_\alpha+\beta\mathbf{P}_\beta+\gamma\mathbf{P}_\gamma+\delta\mathbf{P}_\delta)-\check{\mathbf{w}}^T\mathbf{R}$$
$$\ge \alpha P^*(\mathbf{R}_\alpha)+\beta P^*(\mathbf{R}_\beta)+\gamma P^*(\mathbf{R}_\gamma)+\delta P^*(\mathbf{R}_\delta) \tag{A.3a}$$

$$\ge P^*(\alpha\mathbf{R}_\alpha+\beta\mathbf{R}_\beta+\gamma\mathbf{R}_\gamma+\delta\mathbf{R}_\delta)=\hat{\mathbf{w}}^T\tilde{\mathbf{P}}-\check{\mathbf{w}}^T\mathbf{R}, \tag{A.3b}$$

where the second inequality follows from the convexity of $P^*(\mathbf{R})$ and where $\tilde{\mathbf{P}}$ is the minimal sum-power corresponding to a feasible solution for (3), obtainable by GBL as explained above. Hence, if there exists such a convex combination which meets the sum-power and sum-rate constraints, we can compute a feasible solution to the primal problem in (3) by GBL which has a lower or equal objective. The reverse holds as a primal feasible solution is in $\check{\mathcal{Q}}$. The proof follows from tight duality between the optimization over convex combinations meeting the sum-power and sum-rate constraints, and the dual problem in (5) [24]. □

Appendix B. Solution of the relaxation in (11)

In the following we detail the exact solution of the problem in (11) and use the short notation $\lfloor r^c_u(\mathbf{p}^c)\rfloor_\theta$ and $\lceil r^c_u(\mathbf{p}^c)\rceil_\theta$ to denote the rate rounded up or down to the next integer multiple of the bit-step θ, respectively. The problem in (11) is separable among users i, $(u+1)\le i\le U$. The continuous relaxation of each of those separated problems is convex and given as

$$\underset{r_i\in[0,\hat{\theta}_i^{\text{mod}}]}{\text{minimize}}\ \tilde{f}_i(r_i)=(\hat{w}_i+\nu_i)(2^{r_i}-1)\frac{\Gamma\tilde{N}_i}{H_{ii}}-(\check{w}_i+\lambda_i)r_i, \tag{B.1}$$

where $\hat{\theta}_i^{\text{mod}}=\min\left\{\hat{\theta},\left\lfloor\log_2\left(1+\frac{H_{ii}\hat{p}_i}{\Gamma\tilde{N}_i}\right)\right\rfloor_\theta\right\}$ and we use the fact that the constraints in (11b) hold with equality at optimum to replace the variable p_i. The problem in (B.1) can be given analytically, for each remaining user i, $(u+1)\le i\le U$ independently, using first-order optimality conditions $\partial\tilde{f}_i(r_i)/\partial r_i=0$, leading to

$$r_i^{\text{tmp}}=\min\left\{\log_2\left(\frac{\lambda_u H_{ii}}{(\hat{w}_i+\nu_i)\log(2)\tilde{N}_i\Gamma}\right),\hat{\theta}_i^{\text{mod}}\right\},\quad (u+1)\le i\le U, \tag{B.2}$$

with corresponding power allocation

$$p_i^{\text{tmp}}(r_i^{\text{tmp}})=(2^{r_i^{\text{tmp}}}-1)\frac{\Gamma\tilde{N}_i}{H_{ii}},\quad (u+1)\le i\le U. \tag{B.3}$$

Inserting (B.2) in (B.3) we can interpret $p_i^{tmp}(r_i^{tmp})$ as the water-filling solution [32, Ex. 5.2] for the water-level $(\hat{w}_i+v_i)/((\check{w}_i+\lambda_i)\log(2))$ under bit-cap and PSD mask constraints, respectively. The solution $\overline{r}_i$, $\overline{p}_i$, $(u+1)\le i\le U$, of the discrete problem in (11) can then be computed by rounding the user's rates r_i^{tmp} to one of the two nearest integer multiples of θ with the lower objective, i.e.,

$$[\overline{r}_i,\overline{p}_i]=\underset{\substack{\{r_i\in\{\lfloor r_i^{tmp}\rfloor_\theta,\lceil r_i^{tmp}\rceil_\theta\},\\ p_i=p_i^{tmp}(r_i)\,|\,r_i\le\hat{\theta}_i^{mod}\}}}{\operatorname{argmin}}\{(\hat{w}_i+v_i)p_i-(\check{w}_i+\lambda_i)r_i\}. \tag{B.4}$$

This holds due to the aforementioned convexity and user-independence.

References

[1] K. Leung, L.-C. Wang, Integrated link adaptation and power control to improve error and throughput performance in broadband wireless packet networks, IEEE Transactions on Wireless Communications 1 (4) (2002) 619–629.

[2] M. Chatterjee, H. Lin, S. Das, Rate allocation and admission control for differentiated services in CDMA data networks, IEEE Transactions on Mobile Computing 6 (2) (2007) 179–191.

[3] M. Pischella, J.-C. Belfiore, Weighted sum throughput maximization in multicell OFDMA networks, IEEE Transactions on Vehicular Technology 59 (2) (2010) 896–905.

[4] R. Cendrillon, W. Yu, M. Moonen, J. Verlinden, T. Bostoen, Optimal multiuser spectrum balancing for digital subscriber lines, IEEE Transactions on Communications 54 (5) (2006) 922–933.

[5] M. Wolkerstorfer, D. Statovci, T. Nordström, Robust spectrum management for DMT-based systems, IEEE Transactions on Signal Processing 58 (6) (2010) 3238–3250.

[6] P. Tsiaflakis, J. Vangorp, M. Moonen, J. Verlinden, A low complexity optimal spectrum balancing algorithm for digital subscriber lines, Signal Processing 87 (7) (2007) 1735–1753.

[7] Y. Xu, T. Le-Ngoc, S. Panigrahi, Global concave minimization for optimal spectrum balancing in multi-user DSL networks, IEEE Transactions on Signal Processing 56 (7) (2008) 2875–2885.

[8] P. Tsiaflakis, C. Tan, Y. Yi, M. Chiang, M. Moonen, Optimality certificate of dynamic spectrum management in multi-carrier interference channels, in: IEEE International Symposium on Information Theory 2008 (ISIT'08), Toronto, Canada, 2008, pp. 1298–1302.

[9] L.P. Qian, Y.J. Zhang, J. Huang, MAPEL: achieving global optimality for a non-convex wireless power control problem, IEEE Transactions on Wireless Communications 8 (3) (2009) 1536–1563.

[10] P. Weeraddana, M. Codreanu, M. Latva-aho, A. Ephremides, Resource allocation for cross-layer utility maximization in wireless networks, IEEE Transactions on Vehicular Technology 60 (6) (2011) 2790–2809.

[11] P. Golden, H. Dedieu, K. Jacobsen (Eds.), Fundamentals of DSL Technology, Auerbach Publications, 2006.

[12] J. Campello, Optimal discrete bit loading for multicarrier modulation systems, in: IEEE International Symposium on Information Theory 1998 (ISIT'98), Cambridge, MA, USA, 1998, p. 193.

[13] E. Baccarelli, M. Biagi, Optimal integer bit-loading for multicarrier ADSL systems subject to spectral-compatibility limits, Signal Processing 84 (4) (2004) 729–741.

[14] J. Lee, R. Sonalkar, J. Cioffi, Multi-user discrete bit-loading for DMT-based DSL systems, in: IEEE Global Telecommunications Conference 2002 (GLOBECOM'02), vol. 2, Taipei, Taiwan, China, 2002, pp. 1259–1263.

[15] D. Yu, K. Seong, J. Cioffi, Multiuser discrete bit-loading for digital subscriber lines, in: IEEE International Conference on Communications 2007 (ICC'07), Glasgow, Scotland, 2007, pp. 2755–2760.

[16] M. Wolkerstorfer, T. Nordström, D. Statovci, Energy-Efficient Spectrum Balancing in DSL, Technical Report FTW-TR-2010-003, The Telecommunications Research Center Vienna (FTW), Vienna, Austria, 2010.

[17] N. Sahinidis, BARON—Branch and Reduce Optimization Navigator, Version 4.0, University of Illinois at Urbana, Champaign, 2000.

[18] P. Belotti, Couenne: A User's Manual, Lehigh University, 2009.

[19] H. Ryoo, N. Sahinidis, A branch-and-reduce approach to global optimization, Journal of Global Optimization 8 (2) (1996) 107–138.

[20] P. Belotti, J. Lee, L. Liberti, F. Margot, A. Wächter, Branching and bounds tightening techniques for non-convex MINLP, Optimization Methods and Software 24 (4–5) (2009) 597–634.

[21] R. Cambini, F. Salvi, A branch and reduce approach for solving a class of low rank d.c. programs, Journal of Computational and Applied Mathematics 233 (2009) 492–501.

[22] D. Statovci, T. Nordström, Adaptive resource allocation in multiuser FDD-DMT systems, in: European Signal Processing Conference 2004 (Eusipco'04), Vienna, Austria, 2004, pp. 1213–1216.

[23] R. Yates, A framework for uplink power control in cellular radio systems, IEEE Journal on Selected Areas in Communications 13 (7) (1995) 1341–1347.

[24] M. Wolkerstorfer, J. Jaldén, T. Nordström, Column generation for discrete-rate multi-user and multi-carrier power control, IEEE Transactions on Communications, submitted for publication.

[25] D.P. Bertsekas, Nonlinear Programming, Athena Scientific, 1999.

[26] A.H. Land, A.G. Doig, An automatic method of solving discrete programming problems, Econometrica 28 (3) (1960) 497–520.

[27] M. Wolkerstorfer, T. Nordström, Heuristics for discrete power control — A case-study in multi-carrier DSL networks, in: ALIO/EURO Workshop on Applied Combinatorial Optimization 2011, Porto, Portugal, 2011.

[28] ITU, Asymmetric Digital Subscriber Line Transceivers 2 (ADSL2), Technical Report G.992.3, ITU-T, 2002.

[29] ETSI, Transmission and Multiplexing (TM); Access Transmission Systems on Metallic Access Cables; Very High Speed Digital Subscriber Line (VDSL); Part 1: Functional Requirements, Technical Report TM6 TS 101 270-1, Version 1.3.1, ETSI, 2003.

[30] E. Dolan, J. Moré, Benchmarking optimization software with performance profiles, Mathematical Programming 91 (2) (2002) 201–213.

[31] M. Grötschel, L. Lovász, A. Schrijver, Geometric Algorithms and Combinatorial Optimization, Springer, 1988.

[32] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.